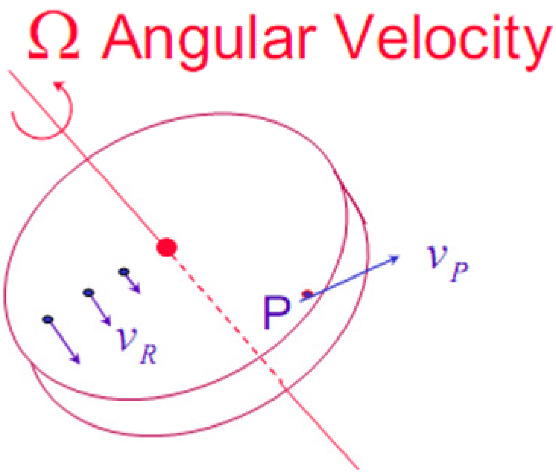
Lecture 06
Velocity Propagation

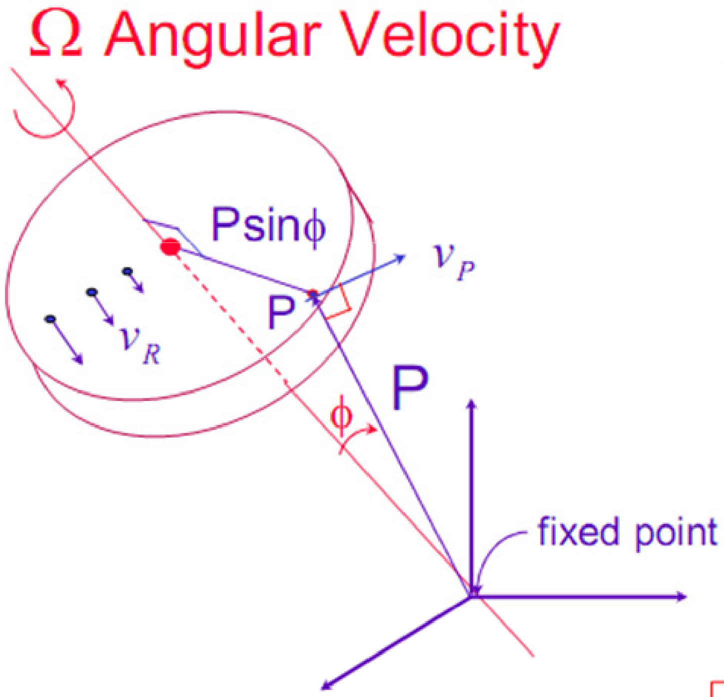
Acknowledgement :
Prof. Oussama Khatib, Robotics Laboratory, Stanford University, USA
Prof. Harry Asada, AI Laboratory, MIT, USA

Rotational Motion



$$\boxed{v_P = ?}$$

Rotational Motion



v_P is proportional to:

- $||\Omega||$
- $||P\sin\phi||$

and

- $v_P \perp \Omega$
- $v_P \perp P$

$$\boxed{v_P = \Omega \times P}$$

Eg: $\Omega = \langle 1,3,2 \rangle$, $P = \langle -2, 0, 1 \rangle$

$\Omega \times P = \langle (3\times1), -(1\times1)+2\times-2, -(3\times-2) \rangle$

$\Omega \times P = \langle 3, -5, 6 \rangle$

Cross Product Operator

$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, b = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$c = a \times b \Rightarrow c = \hat{a}b$$

vectors ⇨ matrices

$a \times \;\Rightarrow\; \hat{a}$: a skew-symmetric matrix

$$c = \hat{a}b = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\boxed{c = \hat{a}b}$$

Cross Product Operator

$$v_P = \Omega \times P \Rightarrow v_P = \hat{\Omega} P$$

$$\Omega \times \Rightarrow \hat{\Omega}$$: a skew-symmetric matrix

$$\Omega = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}; P = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

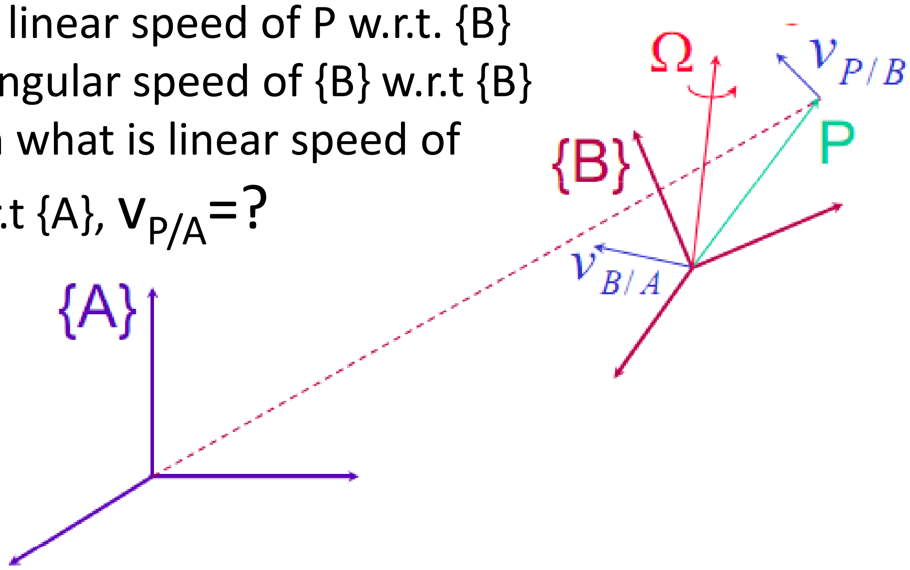
$$v_P = \hat{\Omega} P = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} \cdot \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} \qquad \boxed{v_P = \hat{\Omega} P}$$

Eg: Ω=<1,3,2>, P=<-2, 0, 1>

$$\Omega \times P = \hat{\Omega} P = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}$$

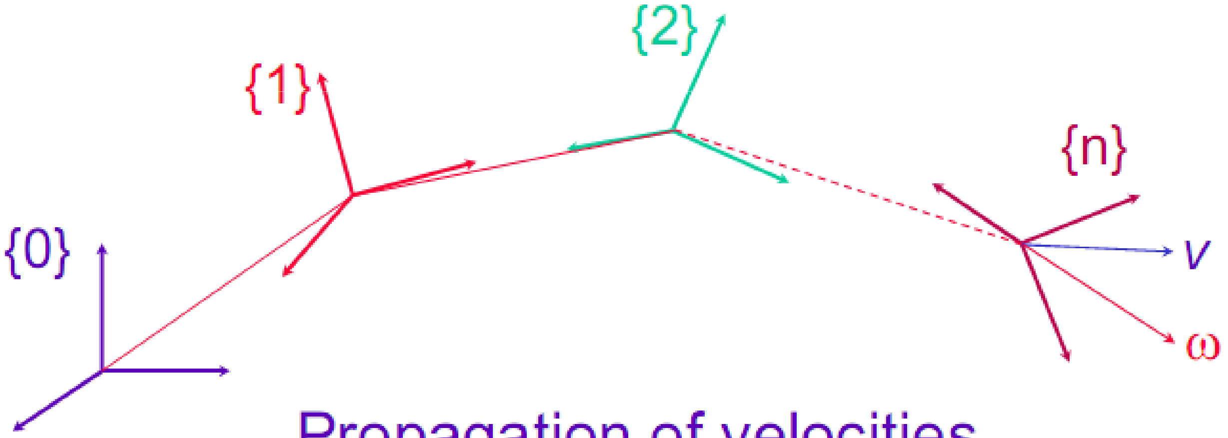
Linear and Angular Velocity

$v_{B/A}$: linear speed of {B} w.r.t. {A}
$v_{P/B}$: linear speed of P w.r.t. {B}
Ω : angular speed of {B} w.r.t {B}
Then what is linear speed of
P w.r.t {A}, $v_{P/A}$=?



Velocity Propagates from Base to End

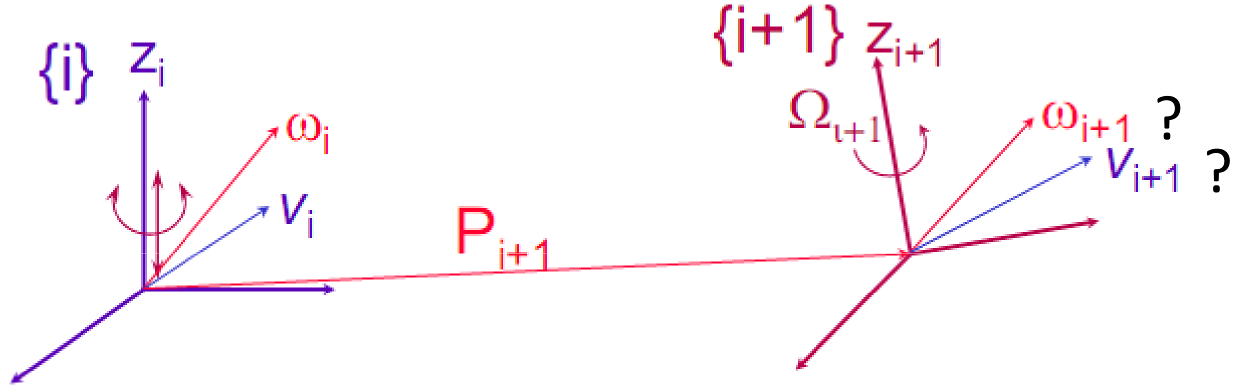
Spatial Mechanisms



Propagation of velocities

$\dot{x}$ — v : linear velocity
— ω : angular velocity

Velocity Propagation from {i} to {i+1}

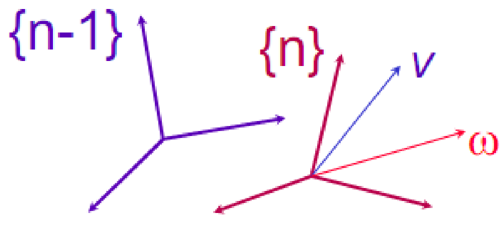


$$^{i+1}\omega_{i+1} = {}^{i+1}_{i}R.{}^{i}\omega_i + \dot{\theta}_{i+1}.{}^{i+1}Z_{i+1}$$

$$^{i+1}v_{i+1} = {}^{i+1}_{i}R.({}^{i}v_i + {}^{i}\omega_i \times {}^{i}P_{i+1}) + \dot{d}_{i+1}.{}^{i+1}Z_{i+1}$$

$\Longrightarrow$ $^{n}\omega_n$ and $^{n}v_n$

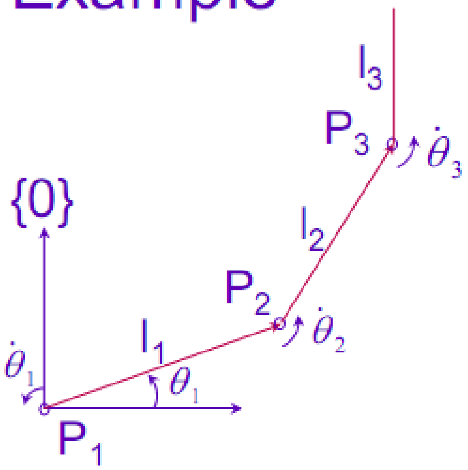
$$\begin{pmatrix} ^{0}v_n \\ ^{0}\omega_n \end{pmatrix} = \begin{pmatrix} ^{0}_{n}R & 0 \\ 0 & ^{0}_{n}R \end{pmatrix} . \begin{pmatrix} ^{n}v_n \\ ^{n}\omega_n \end{pmatrix}$$



Computationally Complex because of frame transformation

Tutorial

Example



Fixed frame of reference {0}

$$v_{i+1} = v_i + \omega_i \times P_{i+1}$$

- $v_{P_1} = 0$ $\quad {}^0\omega_1 = \dot{\theta}_1 . {}^0Z_1$
- $v_{P_2} = v_{P_1} + \omega_1 \times P_2$
- $v_{P_3} = v_{P_2} + \omega_2 \times P_3$

w.r.t. {0}

$${}^0v_{P_2} = 0 + \begin{bmatrix} 0 & -\dot{\theta}_1 & 0 \\ \dot{\theta}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} l_1.c_1 \\ l_1.s_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} . \dot{\theta}_1$$

Computationally efficient because of fixed frame of Ref. {0}

$${}^0\omega_2 = {}^0\omega_1 + {}^0\dot{\theta}_2 \hat{z}_2 = \begin{bmatrix} 0 & 0 & (\dot{\theta}_1 + \dot{\theta}_2) \end{bmatrix}^T$$

$${}^0v_{P_3} = {}^0v_{P_2} + {}^0\omega_2 \times {}^0P_3$$

Angular velocity of P3 w.r.t {0}

2^{nd} link w.r.t {0}

$${}^0v_{P_3} = \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} . \dot{\theta}_1 + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \underbrace{(\dot{\theta}_1 + \dot{\theta}_2)} . {}^0P_3$$

$$\begin{bmatrix} l_2.c_{12} \\ l_2.s_{12} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1.s_1 \\ l_1.c_1 \\ 0 \end{bmatrix} . \dot{\theta}_1 + \begin{bmatrix} -l_2.s_{12} \\ l_2.c_{12} \\ 0 \end{bmatrix} . (\dot{\theta}_1 + \dot{\theta}_2)$$

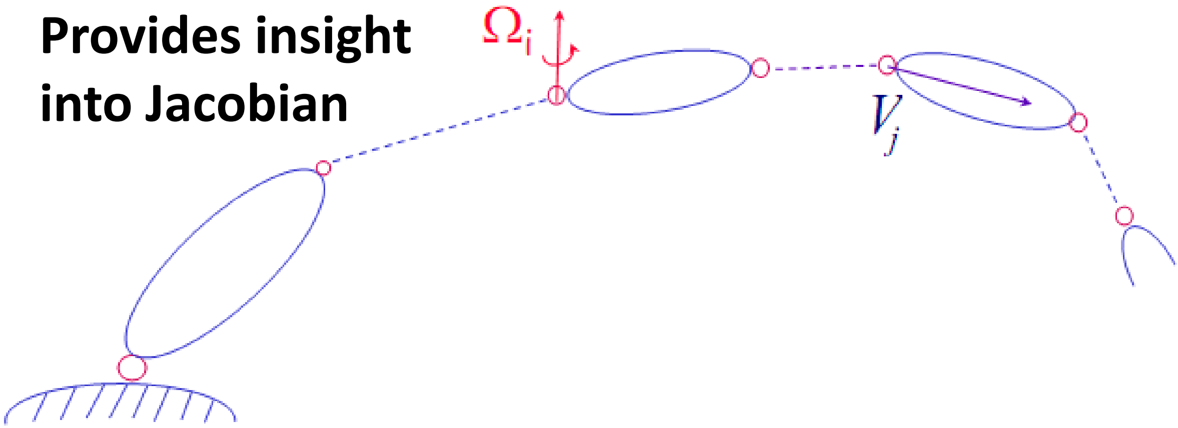
$$ {}^{0}v_{P_3} = \underbrace{\begin{bmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} & 0 \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{J_v} . \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} $$

$$ {}^{0}\omega_3 = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{J_\omega} . \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \Leftarrow {}^{0}\omega_3 = {}^{0}\omega_2 + {}^{0}\dot{\theta}_3 \hat{z}_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} $$

$$ \begin{pmatrix} v \\ \omega \end{pmatrix} = J . \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix} $$

The Jacobian (EXPLICIT FORM)

Provides insight into Jacobian



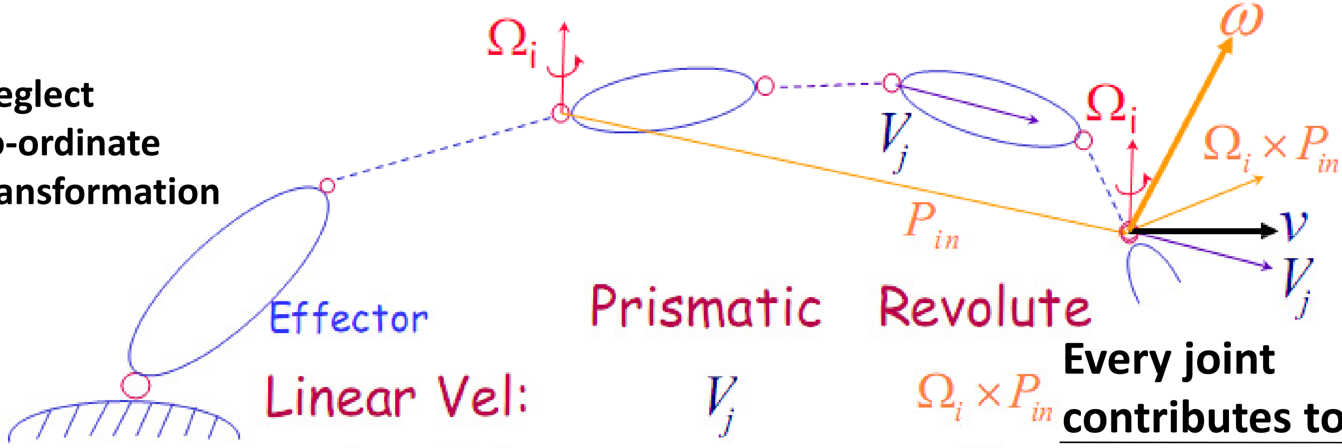
Revolute Joint $\Omega_i = Z_i \dot{q}_i$

Prismatic Joint $V_i = Z_i \dot{q}_i$

Describe how each joint contributes to V and ω of the end-effector

The Jacobian (EXPLICIT FORM)

Neglect co-ordinate transformation



	Prismatic	Revolute
Linear Vel:	V_j	$\Omega_i \times P_{in}$
Angular Vel:	none	Ω_i

Every joint contributes to v

Only R joints contribute to ω

Effector Linear Velocity

$$v = \sum_{i=1}^{n} [\epsilon_i V_i + \bar{\epsilon}_i (\Omega_i \times P_{in})] \quad \Leftarrow \quad V_i = Z_i \dot{q}_i$$

Effector Angular Velocity

$$\omega = \sum_{i=1}^{n} \bar{\epsilon}_i \Omega_i \quad \Leftarrow \quad \Omega_i = Z_i \dot{q}_i$$

Contributes only if the last joint is prismatic

$$v = [\epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n})]\dot{q}_1 + \cdots$$
$$+ [\epsilon_{n-1} Z_{n-1} + \bar{\epsilon}_{n-1} (Z_{n-1} \times P_{(n-1)n})]\dot{q}_{n-1} + \epsilon_n Z_n \dot{q}_n$$

$$v = \begin{bmatrix} \epsilon_1 Z_1 + \bar{\epsilon}_1 (Z_1 \times P_{1n}) & \epsilon_2 Z_2 + \bar{\epsilon}_2 (Z_2 \times P_{2n}) & \cdots \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{v = J_v \dot{q}}$$

There is an easier way to determine J_v using T

$$\omega = \bar{\epsilon}_1 Z_1 \dot{q}_1 + \bar{\epsilon}_2 Z_2 \dot{q}_2 + \cdots + \bar{\epsilon}_n Z_n \dot{q}_n$$

$$\omega = \begin{bmatrix} \bar{\epsilon}_1 Z_1 & \bar{\epsilon}_2 Z_2 & \cdots & \bar{\epsilon}_n Z_n \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$\boxed{\omega = J_\omega \dot{q}}$$

Jacobian of a 6dof manipulator

All rotary manipulator (eg. Puma560)

Without frame transformation

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{bmatrix} z_1 \times P_{16} & z_2 \times P_{26} & z_3 \times P_{36} & z_4 \times P_{46} & z_5 \times P_{56} & z_6 \times P_{66} \\ z_1 & z_3 & z_3 & z_4 & z_5 & z_6 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

With co-ordinate transformation

$$\begin{pmatrix} {}^0v \\ {}^0\omega \end{pmatrix} = \begin{bmatrix} {}^0_1R(z_1 \times P_{16}) & {}^0_2R(z_2 \times P_{26}) & {}^0_3R(z_3 \times P_{36}) & {}^0_4R(z_4 \times P_{46}) & {}^0_5R(z_5 \times P_{56}) & {}^0_6R(z_6 \times P_{66}) = 0 \\ {}^0_1Rz_1 & {}^0_2Rz_3 & {}^0_3Rz_3 & {}^0_4Rz_4 & {}^0_5Rz_5 & {}^0_6Rz_6 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

Jacobian of a 6dof manipulator

Stanford Schinman Arm (RR**P**RRR)

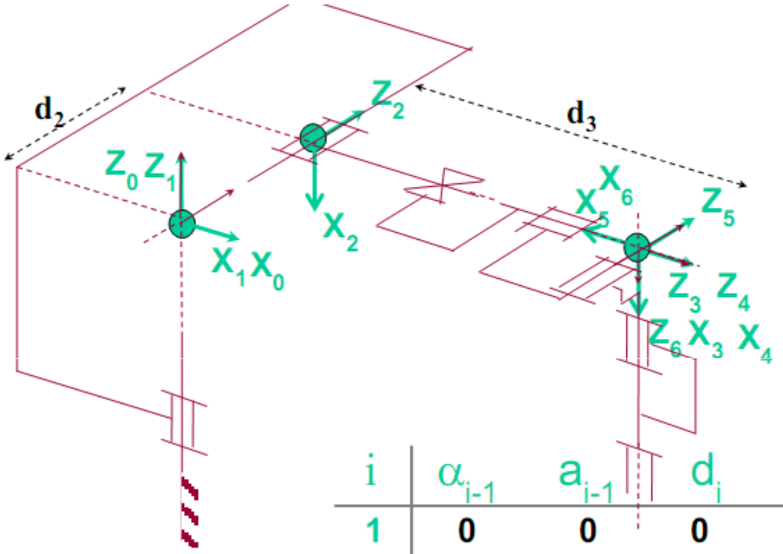
Without frame transformation

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{bmatrix} z_1 \times P_{16} & z_2 \times P_{26} & z_3 & z_4 \times P_{46} = 0 & z_5 \times P_{56} = 0 & z_6 \times P_{66} = 0 \\ z_1 & z_3 & 0 & z_4 & z_5 & z_6 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

With co-ordinate transformation

$$\begin{pmatrix} {}^0v \\ {}^0\omega \end{pmatrix} = \begin{bmatrix} {}^0_1R(z_1 \times P_{16}) & {}^0_2R(z_2 \times P_{26}) & {}^0_3Rz_3 & 0 & 0 & 0 \\ {}^0_1Rz_1 & {}^0_2Rz_2 & 0 & {}^0_4Rz_4 & {}^0_5Rz_5 & {}^0_6Rz_6 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

Stanford Scheinman Arm



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$$

$$J = \begin{array}{cccccc} \mathbf{R} & \mathbf{R} & \mathbf{P} & \mathbf{R} & \mathbf{R} & \mathbf{R} \end{array}$$
$$J = \left(\begin{array}{c|c|c|c|c|c} Z_1 \times P_{13} & Z_2 \times P_{23} & Z_3 & 0 & 0 & 0 \\ \hline Z_1 & Z_2 & 0 & Z_4 & Z_5 & Z_6 \end{array}\right)$$

Matrix J_v (direct differentiation)

$$v = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{x}_P = \underbrace{\frac{\partial x_P}{\partial q_1}}_{3\times 1} . \dot{q}_1 + \underbrace{\frac{\partial x_P}{\partial q_2}}_{3\times 1} . \dot{q}_2 + \cdots + \underbrace{\frac{\partial x_P}{\partial q_n}}_{3\times 1} . \dot{q}_n$$

$\longleftarrow$ 3×n $\longrightarrow$

$$J_v = \begin{pmatrix} \dfrac{\partial x_P}{\partial q_1} & \dfrac{\partial x_P}{\partial q_2} & \cdots & \dfrac{\partial x_P}{\partial q_n} \end{pmatrix}$$

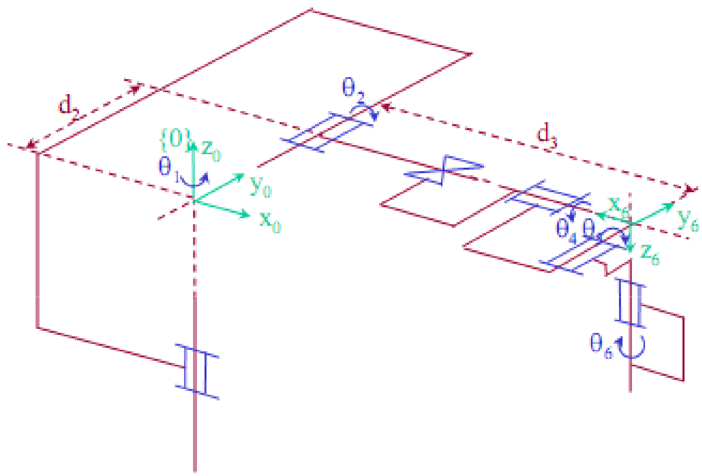
$$\begin{array}{cc} & \begin{array}{cccc} & & & x_p \end{array} \\ {}^0_nT = & \begin{bmatrix} & & & \boxed{\begin{array}{c} x \\ y \\ z \end{array}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

SCA Jacobian from HTMs

$$
\begin{array}{cccccc}
\dfrac{\partial {}^0_6T(:4)}{\partial q_1} & \dfrac{\partial {}^0_6T(:4)}{\partial q_2} & {}^0_3R(:3) & & & \\
 & & {}^0\hat{Z}_3 & & & \\
\mathbf{R} & \mathbf{R} & \mathbf{P} & \mathbf{R} & \mathbf{R} & \mathbf{R}
\end{array}
$$

$$
J = \left(\begin{array}{c:c:c:c:c:c}
Z_1 \times P_{13} & Z_2 \times P_{23} & Z_3 & 0 & 0 & 0 \\ \hdashline
Z_1 & Z_2 & 0 & Z_4 & Z_5 & Z_6
\end{array}\right)
$$

$$
\begin{array}{cccccc}
{}^0\hat{Z}_1 & {}^0\hat{Z}_2 & & {}^0\hat{Z}_4 & {}^0\hat{Z}_5 & {}^0\hat{Z}_6 \\
{}^0_1R(:3) & {}^0_2R(:3) & & {}^0_4R(:3) & {}^0_5R(:3) & {}^0_6R(:3)
\end{array}
$$



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90	0	d_2	θ_2
3	90	0	d_3	0
4	0	0	0	θ_4
5	-90	0	0	θ_5
6	90	0	0	θ_6

$$
{}^{i-1}_{\ \ i}T = \begin{bmatrix}
c\theta_i & -s\theta_i & 0 & a_{i-1} \\
s\theta_i\, c\alpha_{i-1} & c\theta_i\, c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}\, d_i \\
s\theta_i\, s\alpha_{i-1} & c\theta_i\, s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}\, d_i \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

Forward Kinematics: $\quad {}^0_NT = {}^0_1T \; {}^1_2T \; \dots \; {}^{N-1}_{\ \ \ N}T$

$$ {}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ -s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -d_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{4}T = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{4}_{5}T = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_5 & -c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{5}_{6}T = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$${}^{0}_{1}R(:3)$$

$$
{}^{0}_{1}T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$${}^{0}_{2}R(:3)$$

$$
{}^{0}_{2}T = \begin{bmatrix} c_1 c_2 & -c_1 s_2 & -s_1 & -s_1 d_2 \\ s_1 c_2 & -s_1 s_2 & c_1 & c_1 d_2 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}_{3}T = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & c_1 d_3 s_2 - s_1 d_2 \\ s_1 c_2 & c_1 & s_1 s_2 & s_1 d_3 s_2 + c_1 d_2 \\ -s_2 & 0 & c_2 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$${}^{0}_{3}R(:3)$$

$$
{}^{0}_{4}T = \begin{bmatrix} c_1 c_2 c_4 - s_1 s_4 & -c_1 c_2 s_4 - s_1 c_4 & c_1 s_2 & c_1 d_3 s_2 - s_1 d_2 \\ s_1 c_2 c_4 + c_1 s_4 & -s_1 c_2 s_4 + c_1 c_4 & s_1 s_2 & s_1 d_3 s_2 + c_1 d_2 \\ -s_2 c_4 & s_2 s_4 & c_2 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$${}^{0}_{4}R(:3)$$

$$
{}^{0}_{5}T = \begin{bmatrix} X & X & -c_1 c_2 s_4 - s_1 c_4 & c_1 d_3 s_2 - s_1 d_2 \\ X & X & -s_1 c_2 s_4 + c_1 c_4 & s_1 d_3 s_2 + c_1 d_2 \\ X & X & s_2 s_4 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$${}^{0}_{5}R(:3)$$

$$
{}^{0}_{6}T = \begin{bmatrix} X & X & c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 s_5 & c_1 d_3 s_2 - s_1 d_2 \\ X & X & s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 & s_1 d_3 s_2 + c_1 d_2 \\ X & X & -s_2 c_4 s_5 + c_5 c_2 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$${}^{0}_{6}R(:3)$$

Stanford Scheinman Arm Jacobian

$$
{}^{0}J = \begin{pmatrix} \frac{\partial^{0} x_P}{\partial q_1} & \frac{\partial^{0} x_P}{\partial q_2} & \frac{\partial^{0} x_P}{\partial q_3} & 0 & 0 & 0 \\ {}^{0}Z_1 & {}^{0}Z_2 & 0 & {}^{0}Z_4 & {}^{0}Z_5 & {}^{0}Z_6 \end{pmatrix}
$$

$$
\begin{bmatrix}
-c_1 d_2 - s_1 s_2 d_3 & c_1 c_2 d_3 & c_1 s_2 & 0 & 0 & 0 \\
-s_1 d_2 + c_1 s_2 d_3 & s_1 c_2 d_3 & s_1 s_2 & 0 & 0 & 0 \\
0 & -s_2 d_3 & c_2 & 0 & 0 & 0 \\
0 & -s_1 & 0 & c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\
0 & c_1 & 0 & s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\
1 & 0 & 0 & c_2 & s_2 s_4 & -s_2 c_4 s_5 + c_5 c_2
\end{bmatrix}
$$

Jacobian in a Frame

Vector Representation

$$
J = \begin{pmatrix} \frac{\partial x_P}{\partial q_1} & \frac{\partial x_P}{\partial q_2} & \cdots & \frac{\partial x_P}{\partial q_n} \\ \bar{\epsilon}_1 . Z_1 & \bar{\epsilon}_2 . Z_2 & \cdots & \bar{\epsilon}_n . Z_n \end{pmatrix}
$$

In {0}

$$
{}^{0}J = \begin{pmatrix} \frac{\partial^{0} x_P}{\partial q_1} & \frac{\partial^{0} x_P}{\partial q_2} & \cdots & \frac{\partial^{0} x_P}{\partial q_n} \\ \bar{\epsilon}_1 . {}^{0}Z_1 & \bar{\epsilon}_2 . {}^{0}Z_2 & \cdots & \bar{\epsilon}_n . {}^{0}Z_n \end{pmatrix}
$$

J in Frame {0}

$${}^{0}Z_i = {}^{0}_{i}R \; {}^{i}Z_i; \quad {}^{i}Z_i = Z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

3rd colomn in ${}^{0}_{i}R$

$${}^{0}J = \begin{pmatrix} \frac{\partial}{\partial q_1}({}^{0}x_P) & \frac{\partial}{\partial q_2}({}^{0}x_P) & \cdots & \frac{\partial}{\partial q_n}({}^{0}x_P) \\ \overline{\in}_1.({}^{0}_{1}R.Z) & \overline{\in}_2.({}^{0}_{2}R.Z) & \cdots & \overline{\in}_n.({}^{0}_{n}R.Z) \end{pmatrix}$$

3rd colomn in ${}^{0}_{1}R$ **3rd colomn in** ${}^{0}_{2}R$ **3rd colomn in** ${}^{0}_{n}R$